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The growth of three-dimensional disturbances in inviscid flows

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It is shown by linear inviscid theory that the most highly amplified instability mode in an incompressible plane parallel flow is a two-dimensional one for either temporally or spatially growing waves.

1. Introduction

Squire (1933) proved that the two-dimensional mode was the first to become unstable in any plane parallel flow. He showed how the stability of an oblique temporally growing wave depended solely on the velocity profile in the direction normal to the wave front, the eigenvalues being related to those of a two-dimensional mode at a lower Reynolds number. The question as to whether or not the most highly amplified wave is plane or oblique is more difficult to answer. When the stability is controlled by viscosity (see Watson 1960 and Michael 1961), it is necessary to have some knowledge of the eigenvalues for all Reynolds numbers and no general statements can be made. However, for very unstable flows, like jets and wakes, viscosity has virtually no influence on the eigenvalues at quite moderate Reynolds numbers and the inviscid form of Squire's transformation can be used. This allows any three-dimensional wave to be related to a two-dimensional one. For temporally growing waves it is quite simple to show that the most highly amplified mode is two-dimensional, but in the physically more important case of spatial modes it is not obvious that the same rule applies. Recently Michalke (1969) suggested that a general statement cannot be made for spatially growing modes and he has resorted to the numerical evaluation of growth rates to show that in the case of a free shear layer the most amplified wave is indeed the two-dimensional one.

The present treatment considers the wave-packet generated by an isolated pulse input. Since such a motion contains the sum of all possible modes it is surely only necessary to find the mode associated with the maximum amplification of the packet to obtain the most highly amplified wave.

2. Analysis

Following Gaster & Davey (1968) we can link the oblique wave

$$\exp\{i(ax+bz-\omega t)\}$$

with the two-dimensional wave

$$\exp\{i(\alpha x-\beta t)\},$$

by putting $\alpha^2 = a^2 + b^2$ and $a\omega = \alpha\beta$ and noting that the eigenvalue relationships are

$$\omega = aE(a^2+b^2)^{\frac{1}{2}} \tag{1}$$

and

$$\beta = \alpha E(\alpha). \tag{2}$$

[These relations are valid only when $(a/\alpha)_r > 0$ and $\beta_i \neq 0$.]

α and a are wave-numbers in the x direction along the stream, b is a spanwise wave-number in the z direction and ω and β are frequency parameters.

The asymptotic form of the motion resulting from a pulse disturbance was shown to be of the form

$$\sim \exp\left\{i\left[\left(\alpha\frac{x}{t}-\beta\right)^2+\alpha^2\left(\frac{z}{t}\right)^2\right]^{\frac{1}{2}}t\right\}, \tag{3}$$

where

$$\frac{x}{t} = \frac{\beta}{\alpha}+\frac{\alpha^2-b^2}{\alpha^2}\left[\frac{d\beta}{d\alpha}-\frac{\beta}{\alpha}\right] \quad \text{and} \quad \frac{z}{t} = \frac{b(\alpha^2-b^2)^{\frac{1}{2}}}{\alpha^2}\left[\frac{d\beta}{d\alpha}-\frac{\beta}{\alpha}\right]. \tag{4}$$

Equations (4) reduce to

$$\left(\frac{x}{t}-\frac{\beta}{\alpha}\right)\left(\frac{x}{t}-\frac{d\beta}{d\alpha}\right)+\left(\frac{z}{t}\right)^2 = 0. \tag{5}$$

Equations (2) and (5) enable α and β to be determined in terms of the real quantities x/t and z/t, and the motion in the physical plane given by (3) can therefore also be found as a function of spatial position.

Putting

$$Q = \left[\left(\alpha\frac{x}{t}-\beta\right)^2+\alpha^2\left(\frac{z}{t}\right)^2\right]^{\frac{1}{2}},$$

we have the asymptotic solution in the form

$$\exp\{iQt\},$$

and the point of maximum temporal amplification is found by equating the derivative of Q_i to zero in the physical plane:

$$\frac{dQ}{d(x/t)} = \frac{1}{Q}\left[\left(\alpha\frac{x}{t}-\beta\right)\left(\alpha+\frac{x}{t}\frac{d\alpha}{d(x/t)}-\frac{d\beta}{d(x/t)}\right)+\alpha\left(\frac{z}{t}\right)^2\frac{d\alpha}{d(x/t)}\right]. \tag{6}$$

With (5) this reduces to

$$\alpha\left(\alpha\frac{x}{t}-\beta\right)\Big/Q. \tag{7}$$

Similarly

$$\frac{dQ}{d(z/t)} = \frac{\alpha(z/t)}{Q}. \tag{8}$$

Equating the imaginary parts of (7) and (8) to zero we get the condition that $\alpha(\alpha(x/t)-\beta)/Q$ and $\alpha(z/t)/Q$ must both be real. Since the sum of the squares of these real quantities reduces to α^2 it is clear that α must also be real. Apart from the trivial solution with Q real, which does not lead to a maximum, we find that z/t must be zero. Equation (4) implies that this instability mode has zero b and is two-dimensional. Furthermore, for zero z/t we have $x/t = d\beta/d\alpha$, or $x/t = \partial\beta_r/\partial\alpha_r$ and $\partial\beta_i/\partial\alpha_r = 0$, so that the mode is the temporally growing wave with the maximum value of β_i.

We can repeat the above procedure to find the most amplified spatial mode. Putting

$$P = \left[\left(\alpha - \beta\frac{t}{x}\right)^2 + \alpha^2\left(\frac{z}{x}\right)^2\right]^{\frac{1}{2}},$$

we have (3) in the form

$$\exp\{iPx\},$$

and the position of the spatial maximum is given by making $dP/d(t/x)$ and $dP/d(z/x)$ real. That is

$$-\frac{\beta(\alpha-\beta(t/x))}{P} \quad \text{and} \quad \frac{\alpha(z/x)}{P}$$

must be real. It is not difficult to show that the only maximum occurs when β is real and z/x is zero. Therefore the wave which produces the most rapidly growing region of the wave-packet must be the two-dimensional spatial mode with the largest amplification factor.

3. Conclusions

It has been shown that the most highly amplified region of an isolated wave-packet occurs on the x axis, whether one considers temporal or spatial growth. It follows from this that the most highly amplified waves are always two-dimensional.

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